North Sydney Girls High School

2023 **HSC TRIAL EXAMINATION**

# Mathematics Extension 1

**General Instructions**

- Reading Time – 10 minutes
- Working Time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

**Total marks: 70**

**Section I – 10 marks** (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II – 60 marks** (pages 8 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

**NAME:**____________________________ **TEACHER:**____________________________

**STUDENT NUMBER:**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

| Question | 1-10 | 11 | 12 | 13 | 14 | Total |
|---|---|---|---|---|---|---|
| Mark | /10 | /15 | /15 | /16 | /14 | /70 |

## Section I

**10 marks**
**Attempt Questions 1-10**
**Allow about 15 minutes for this section**

**Use the multiple choice answer sheet for Questions 1-10.**

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**1** Which of the following is the derivative of $y = \tan^{-1} 2x$?

A. $2\sec^2 2x$

B. $-2\tan^{-2} 2x$

C. $\dfrac{1}{1+4x^2}$

D. $\dfrac{2}{1+4x^2}$

**2** The polynomial $P(x) = 6x^3 - 13x^2 + ax + 24$ is divisible by $(x-2)$. What is the remainder when $P(x)$ is divided by $(x+1)$?

A. $-10$

B. $-5$

C. 7

D. 15

**3** Which of the following is equal to $2\sin 3x \cos 4x$?

A. $\sin 7x - \sin x$

B. $\sin 7x + \sin x$

C. $\cos 7x - \cos x$

D. $\cos x - \cos 7x$

**4** Given that $t = \tan\dfrac{\theta}{2}$, which of the following expressions is equivalent to $\dfrac{\sin\theta + 1}{\cos\theta}$?

A. $\dfrac{1}{t}$

B. $\dfrac{1+t}{1-t}$

C. $\dfrac{\left(1+t^2\right)^2}{1-t^2}$

D. $\dfrac{1+2t-t^2}{1+t^2}$

**5** In how many ways can all the letters of the word STATISTICS be placed in a line with the three Ts together?

A. $\dfrac{10!}{2!3!3!}$

B. $\dfrac{8!}{2!3!3!}$

C. $\dfrac{8!}{2!3!}$

D. $\dfrac{8!}{2!}$

**6** Consider the polynomial $P(x) = x^3 - 3x^2 - 9x + 27$. Which of the following is a root of multiplicity 2 of $P(x)$?

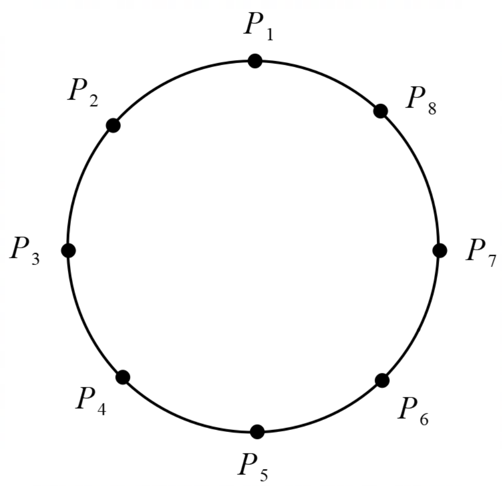
A. 3

B. 1

C. $-1$

D. $-3$

**7** Eight points are arranged in order around a circle as shown below. In a game, Matthew selects 5 points to draw a pentagon. How many games must Matthew play to guarantee that at least 2 pentagons drawn share the same vertices?



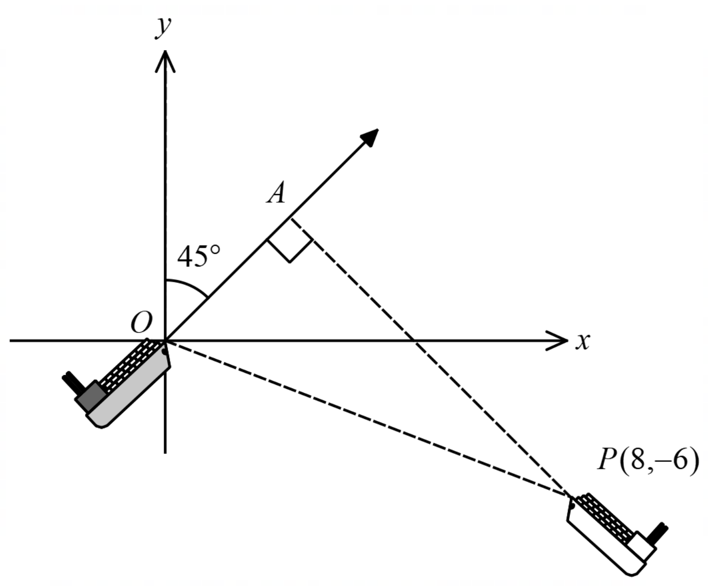

A. 2

B. 12

C. 41

D. 57

**8** A police patrol boat is initially at point $P$ with a position vector $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$. The patrol boat spots a suspicious passenger boat heading from $O$ on a bearing of $045°\mathrm{T}$.




The police patrol boat wants to intercept the passenger boat in the shortest distance as shown by the path $PA$.

Which of the following is the police patrol boat's displacement vector $\overrightarrow{PA}$?

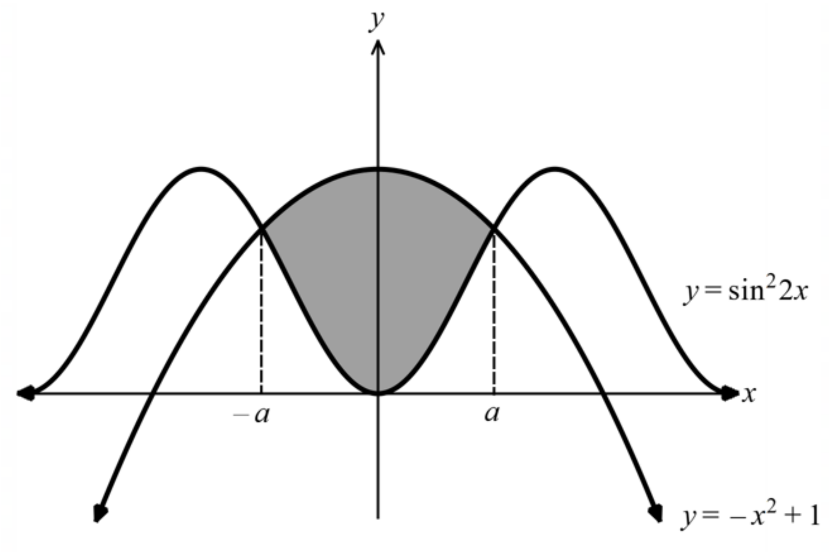
A. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

B. $\begin{pmatrix} -7 \\ 7 \end{pmatrix}$

C. $\begin{pmatrix} -5 \\ 9 \end{pmatrix}$

D. $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

**9** Which of the following integrals correctly represents the area of the shaded region below?



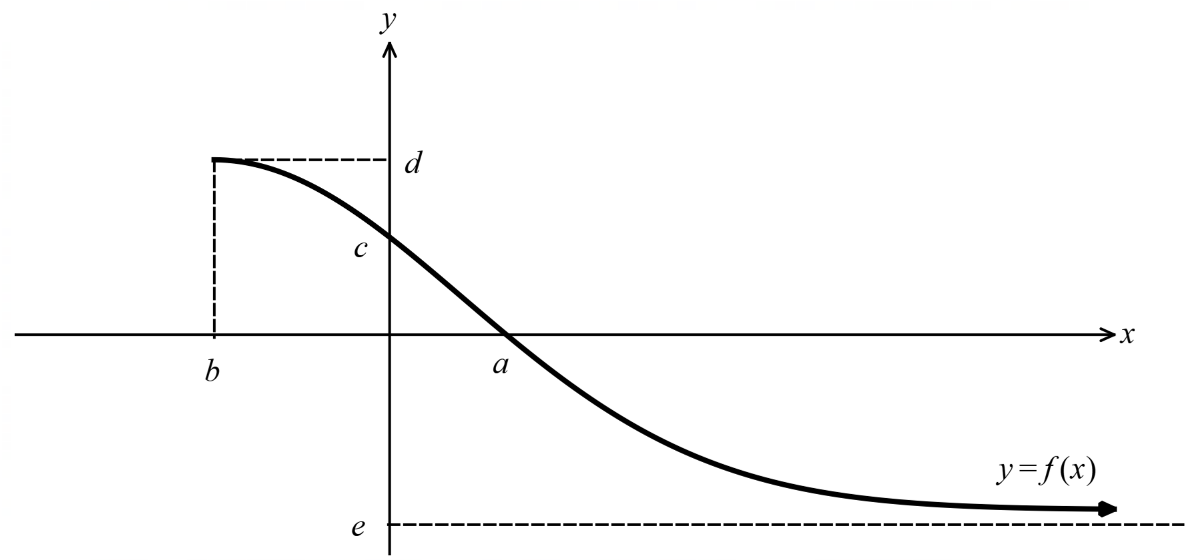

A. $\displaystyle\int_{-a}^{a}\left(x^2 - \frac{1}{2}\cos 4x - \frac{1}{2}\right)dx$

B. $\displaystyle\int_{-a}^{a}\left(-x^2 + \frac{1}{2}\cos 2x + \frac{1}{2}\right)dx$

C. $\displaystyle 2\int_{0}^{a}\left(-x^2 + \frac{1}{2}\cos 4x + \frac{1}{2}\right)dx$

D. $\displaystyle 2\int_{0}^{a}\left(-x^2 + \frac{1}{2}\cos 4x - \frac{1}{2}\right)dx$

**10** The graph of $y = f(x)$ is given below.




The function $g(x)$ is defined as $g(x) = f^{-1}(x)$. Which of the following gives the range of $g(|x|)$?

A. $[b, \infty)$

B. $[b, a]$

C. $[c, d]$

D. $(e, d]$

**End of Section I**

## Section II

**60 marks**
**Attempt Questions 11-14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet

(a) For the vectors $\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, evaluate each of the following.

(i) $\underset{\sim}{u} - 2\underset{\sim}{v}$ **1**

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

(b) Solve $\dfrac{4-x}{x-1} \le 6$. **3**

(c) By using the compound angle formulae, find the exact value of $\tan 15°$. **3**

(d) Eight people are to be seated at a round table.

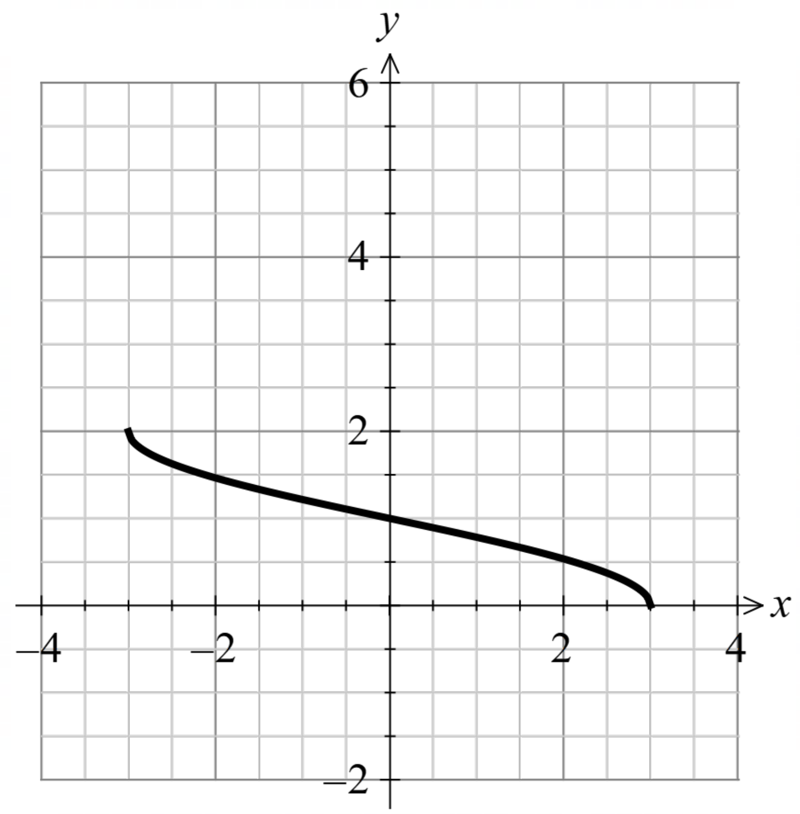
(i) How many seating arrangements are possible? **1**

(ii) Patrick wishes to sit between two of his friends, Jack and Holly. How many seating arrangements are now possible? **2**

**Question 11 continues on page 9**

**<u>Answer Question 11 (e) on the separate Response Sheet.</u>**

(e) The graph of $y = f(x)$ is shown on the separate Response Sheet for Question 11 (e).




(i) Determine the equation of $f(x)$, given that it is in the form $f(x) = a\cos^{-1} bx$. **2**

**<u>Write your answer on the separate Response Sheet</u>**.

(ii) **<u>On the separate Response Sheet,</u>** sketch the graph of $y = \dfrac{1}{f(x)}$, clearly showing all important features. **2**

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet

(a) Find $\int (1+\cos\theta)^2 \, d\theta$. **3**

(b) (i) Find the coefficient of $x^k$ in the expansion of $(2+3x)^6$. **1**

(ii) Hence, or otherwise, find the coefficient of $x^3$ in the expansion of $(2+3x)^6(1-4x^2)$. **2**

(c) A curve is defined in parametric form by the equations $x=\sqrt{t+1}$ and $y=\ln t$.

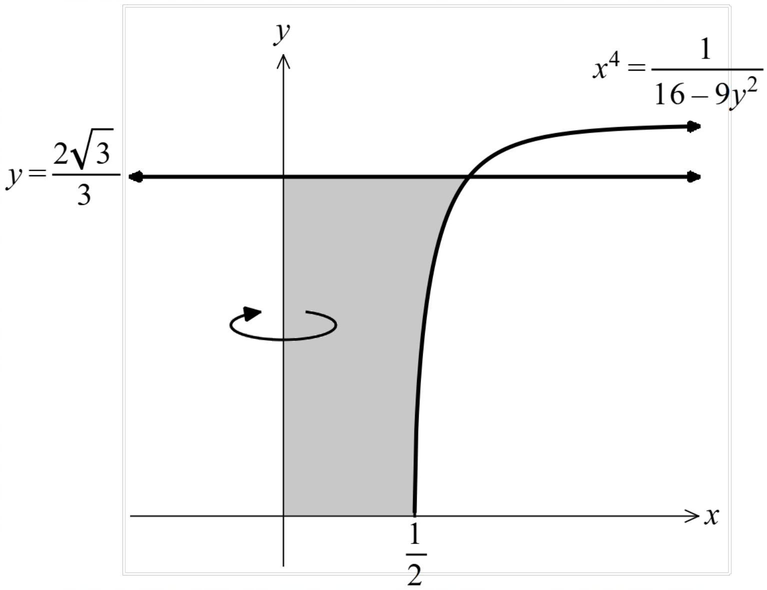
(i) Write the equation of the curve in Cartesian form. **1**

(ii) Sketch the curve clearly, showing all important features. **2**

(d) Use the substitution $u^2=9-x$ to find $\int_0^5 \frac{x}{\sqrt{9-x}}\, dx$. **3**

**Question 12 continues on page 11**

(e) A concrete block is formed by rotating the shaded region bounded by the graph of $x^4 = \dfrac{1}{16-9y^2}$, the $x$ axis, the $y$ axis and the line $y = \dfrac{2\sqrt{3}}{3}$ about the $y$ axis. **3**




Find the exact volume of the solid formed.

**End of Question 12**

**Question 13** (16 marks) Use a SEPARATE writing booklet

(a) Let $f(x) = \sin^{-1} x + \cos^{-1} x$.

(i) By differentiating $f(x)$, or otherwise, explain why $f(x) = \dfrac{\pi}{2}$. **2**

(ii) Hence, sketch the graph of $y = f(x)$. **1**

(b) Let $f(x) = 18x^2 + 3px - 2q$ and $g(x) = 18x^2 + 3qx - 2p$, where $p$, $q$ are distinct real numbers. $\alpha, \beta$ are the roots of the equation $f(x) = 0$ and $\alpha, \gamma$ are the roots of the equation $g(x) = 0$.

(i) By first finding the value of $\alpha$, show that $p + q = 4$. **3**

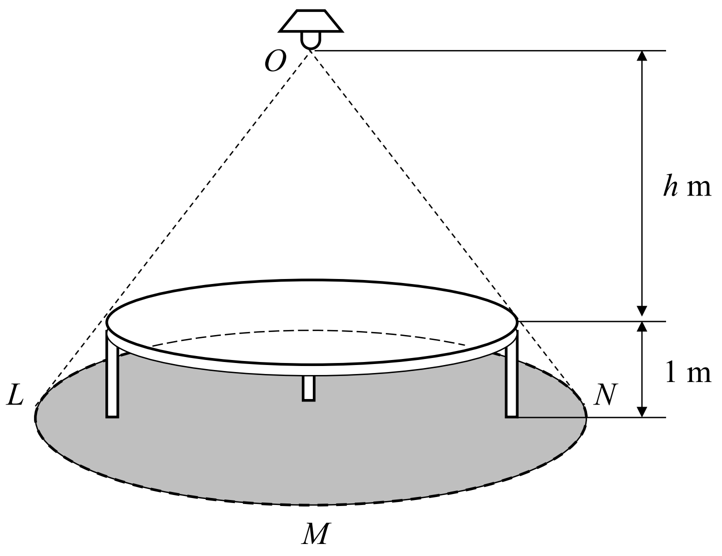
(ii) Hence, or otherwise, express $\beta$ and $\gamma$ in terms of $p$. **2**

(c) Use Mathematical Induction to prove that $3^{3n} + 2^{n+2}$ is divisible by 5 for all integers $n \geq 1$. **3**

**Question 13 continues on page 13**

(d) A small lamp $O$ is placed $h$ m above the centre of a round table of radius 3 m and height of 1 m. The lamp casts a circular shadow $LMN$ of the table on the ground as shown in the diagram below.

At any given time, let $h$ be the height, in metres, of the lamp above the table and let $A$ be the area of the shadow in $\mathrm{m}^2$.



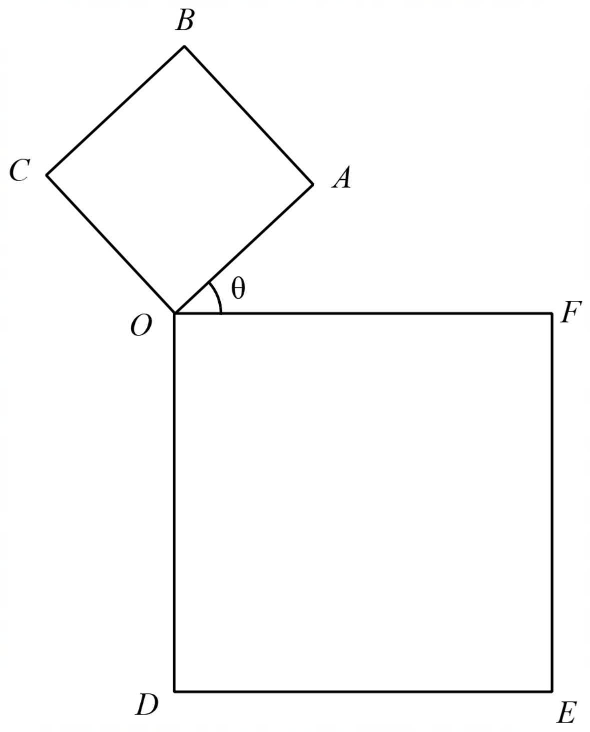

(i) Show that $A = 9\pi\left(1+\frac{1}{h}\right)^2$. **2**

(ii) If the lamp is lowered vertically at a constant rate of $\frac{3}{100}$ m/s, find the exact rate of change of $A$ with respect to time when the lamp is 3 m above the table. **3**

**End of Question 13**

**Question 14** (14 marks) Use a SEPARATE writing booklet

(a) The diagram below shows two squares $OABC$ and $ODEF$ with $OA = 1$, $OD = 2$ and $\angle AOF = \theta$, where $0° < \theta < 90°$. Let $\overrightarrow{OF} = 2\underset{\sim}{i}$, $\overrightarrow{OD} = -2\underset{\sim}{j}$ and $\overrightarrow{OA} = \cos\theta\,\underset{\sim}{i} + \sin\theta\,\underset{\sim}{j}$.




(i) Explain why $\overrightarrow{OC} = -\sin\theta\,\underset{\sim}{i} + \cos\theta\,\underset{\sim}{j}$. **1**

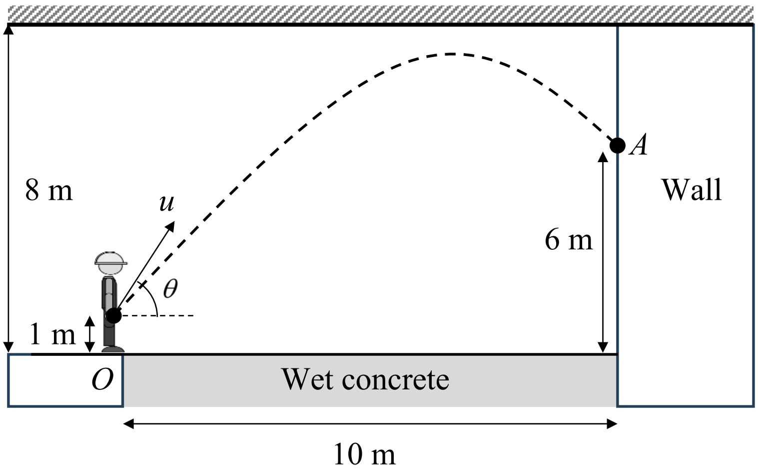
(ii) Use vector methods to show that if the points $B$, $A$ and $E$ are collinear, $\cos\theta - \sin\theta = \frac{1}{2}$. **3**

(iii) By first expressing $\cos\theta - \sin\theta$ in the form of $R\cos(\theta + \alpha)$ where $R > 0$ and $0 < \alpha < 90°$, find the value(s) of $\theta$ such that points $B$, $A$ and $E$ are collinear. Give your answer(s) correct to the nearest degree. Show all working. **3**

**Question 14 continues on page 15**

(b) A construction worker wants to anchor a cable at point $A$ on a wall inside a room. The room is 8 m high and the anchor point for the cable at $A$ is 6 m above the ground on the wall. A 10 m patch of wet concrete prevents the worker from getting close to the wall as shown in the diagram below.

In order to anchor the cable to the wall, the worker needs to launch the cable hook from his hand 1 m above ground at a speed of $u$ m/s and an angle of $\theta°$ to the horizontal. The acceleration due to gravity is $g$ m/s$^2$.




Taking the position of the worker's foot as the origin, the position of the cable hook at time $t$ seconds after it is launched is given by:

$$\vec{r} = \begin{pmatrix} ut\cos\theta \\ -\frac{gt^2}{2} + ut\sin\theta + 1 \end{pmatrix} \qquad \text{(Do NOT prove this)}$$

(i) Show that for the cable hook to avoid hitting the ceiling, $\sin^2\theta < \frac{14g}{u^2}$. **3**

The worker launches the cable hook at 15 m/s. Assuming that $g = 10$ m/s$^2$, the Cartesian equation of the cable hook is given by:

$$y = -\frac{x^2}{45}\sec^2\theta + x\tan\theta + 1 \qquad \text{(Do NOT prove this)}$$

(ii) Find all possible angle(s) that the worker can launch the cable hook to reach the anchor point. Round your answer(s) to the nearest degree. **4**

**End of paper**

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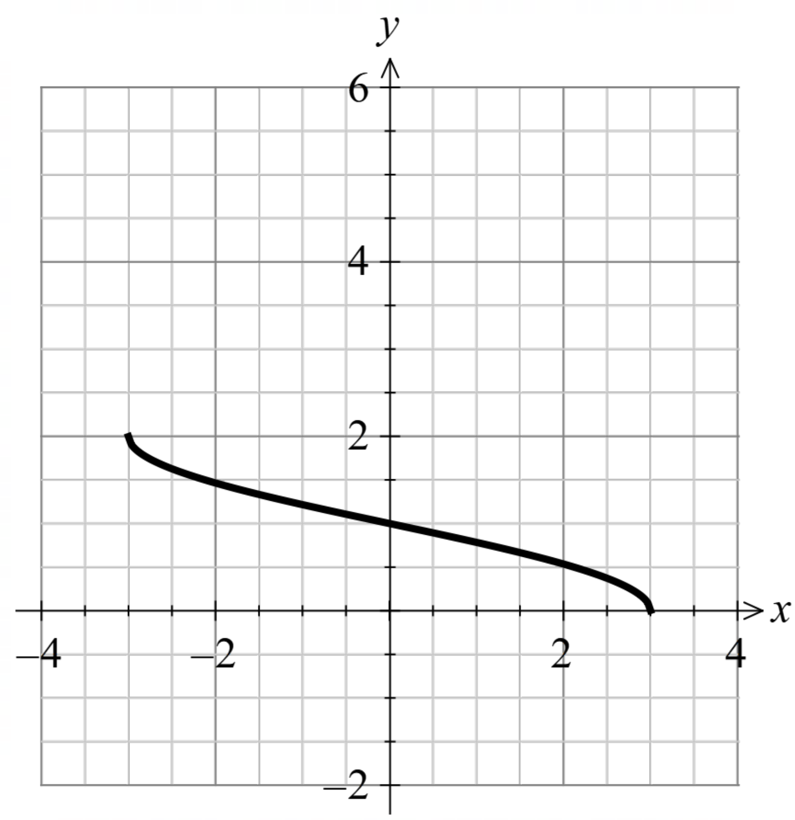
Student Number:________________

## Question 11 (e) – Response Sheet

11 (e) The graph of $f(x)$ is shown below.




(i) Determine the equation of $f\left(x\right)$ given that it is in the form $f\left(x\right)=a\cos^{-1}bx$. **2**
Write your answer below.

$f\left(x\right)=$ ______________________________

(ii) On the same grid above, sketch the graph of $y=\dfrac{1}{f\left(x\right)}$, clearly **2**

showing all important features.

**<u>Place this sheet inside your Question 11 answer booklet.</u>**

North Sydney Girls High School

| 2023 | **HSC TRIAL EXAMINATION** |
|---|---|

# Mathematics Extension 1

**General Instructions**

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- Working Time – 2 hours
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**Total marks: 70**

**Section I – 10 marks** (pages 2 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II – 60 marks** (pages 8 – 15)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

**NAME:**______________________________ **TEACHER:**____________________________

**STUDENT NUMBER:**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

| Question | 1-10 | 11 | 12 | 13 | 14 | Total |
|---|---|---|---|---|---|---|
| Mark | /10 | /15 | /15 | /16 | /14 | /70 |

## Section I

**10 marks**
**Attempt Questions 1-10**
**Allow about 15 minutes for this section**

**Use the multiple choice answer sheet for Questions 1-10.**

---

**1** Which of the following is the derivative of $y = \tan^{-1} 2x$?

A. $2\sec^2 2x$

B. $-2\tan^{-2} 2x$

C. $\dfrac{1}{1+4x^2}$

D. $\dfrac{2}{1+4x^2}$

$$\frac{dy}{dx} = \frac{1}{1+(2x)^2} \times 2 = \frac{2}{1+4x^2}$$

**2** The polynomial $P(x) = 6x^3 - 13x^2 + ax + 24$ is divisible by $(x-2)$. What is the remainder when $P(x)$ is divided by $(x+1)$?

A. $-10$

B. $-5$

C. $7$

D. $15$

$$P(2) = 0$$
$$6\times 2^3 - 13\times 2^2 + a\times 2 + 24 = 0$$
$$a = -10$$
$$P(-1) = 6\times(-1)^3 - 13\times(-1)^2 - 10\times -1 + 24 = 15$$

**3** Which of the following is equal to $2\sin 3x \cos 4x$?

A. $\sin 7x - \sin x$

B. $\sin 7x + \sin x$

C. $\cos 7x - \cos x$

D. $\cos x - \cos 7x$

$$2\sin 3x\cos 4x = \sin(3x+4x) + \sin(3x-4x) = \sin 7x + \sin(-x) = \sin 7x - \sin x$$

**4** Given that $t = \tan\frac{\theta}{2}$, which of the following expressions is equivalent to $\frac{\sin\theta + 1}{\cos\theta}$?

A. $\frac{1}{t}$

B. $\frac{1+t}{1-t}$

C. $\frac{(1+t^2)^2}{1-t^2}$

D. $\frac{1+2t-t^2}{1+t^2}$

$$\frac{\sin\theta+1}{\cos\theta} = \frac{\frac{2t}{1+t^2}+1}{\frac{1-t^2}{1+t^2}} = \frac{2t+1+t^2}{1-t^2} = \frac{(1+t)^2}{(1-t)(1+t)} = \frac{1+t}{1-t}$$

**5** In how many ways can all the letters of the word STATISTICS be placed in a line with the three Ts together?

A. $\dfrac{10!}{2!3!3!}$

B. $\dfrac{8!}{2!3!3!}$

C. $\dfrac{8!}{2!3!}$

D. $\dfrac{8!}{2!}$

Grouping the Ts together gives 8 elements to permute. Within the 8 elements to permute, the repeating elements are: 3 Ss and 2 Is. This gives the number of permutations: $\dfrac{8!}{2!3!}$

**6** Consider the polynomial $P(x) = x^3 - 3x^2 - 9x + 27$. Which of the following is a root of multiplicity 2 of $P(x)$?

A. 3

B. 1

C. $-1$

D. $-3$

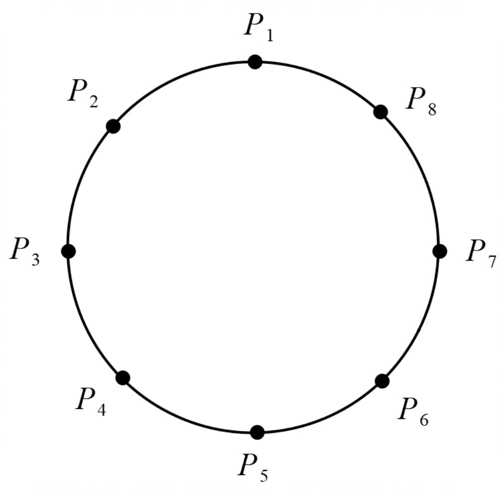
$$P'(x) = 3x^2 - 6x - 9$$
$$P'(x) = 0$$
$$3x^2 - 6x - 9 = 0$$
$$(x-3)(x+1) = 0$$
$$x = -1, 3$$

$P(3) = 3^3 - 3\times 3^2 - 9\times 3 + 27 = 0$

$\therefore P(3) = P'(3) = 0$

$x = 3$ is a root of multiplicity 2 of $P(x)$

**7** Eight points are arranged in order around a circle as shown below. In a game, Matthew selects 5 points to draw a pentagon. How many games must Matthew play to guarantee that at least 2 pentagons drawn share the same vertices?




A. 2

B. 12

C. 41

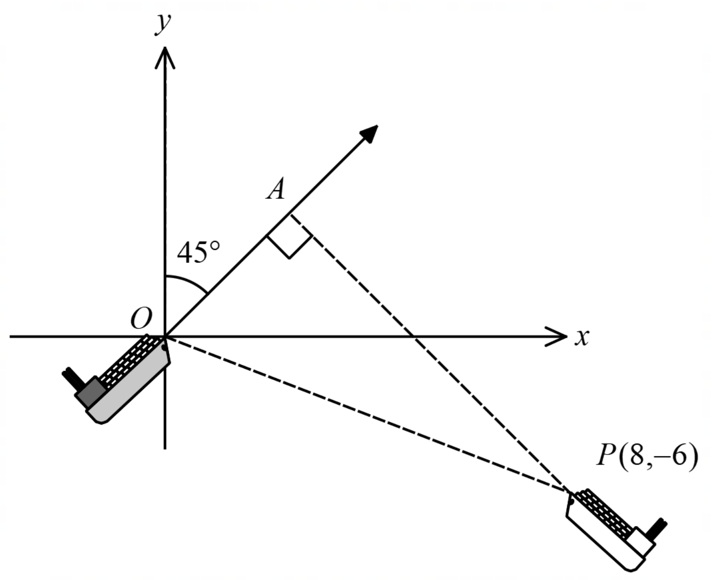
D. 57

Number of pentagons that can be formed without any of them sharing the same vertices:

$$^{8}C_{5} = 56$$

Therefore, the 57th pentagon formed must share the same vertices with one of the 56 pentagons.

**8** A police patrol boat is initially at point $P$ with a position vector $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$. The patrol boat spots a suspicious passenger boat heading from $O$ on a bearing of $045°\text{T}$.




The police patrol boat wants to intercept the passenger boat in the shortest distance as shown by the path $PA$.

Which of the following is the police patrol boat's displacement vector $\overrightarrow{PA}$?

A. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

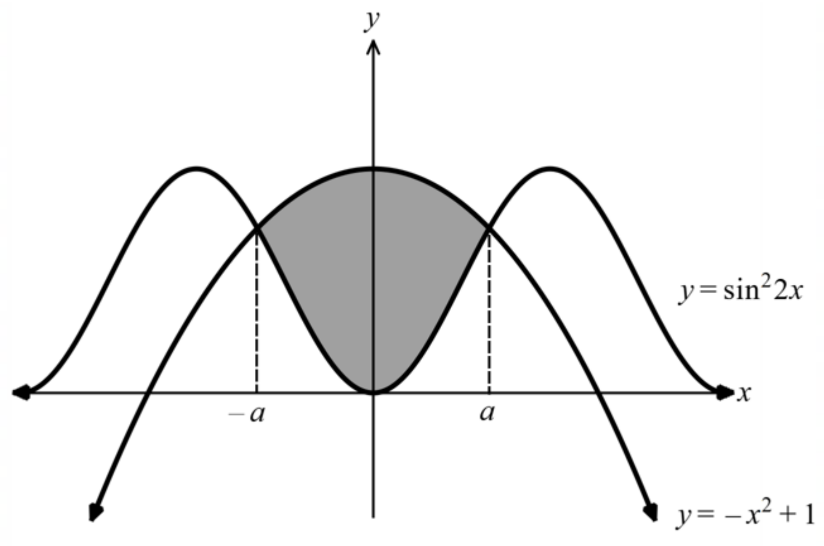
B. $\begin{pmatrix} -7 \\ 7 \end{pmatrix}$

C. $\begin{pmatrix} -5 \\ 9 \end{pmatrix}$

D. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Consider a vector in the direction of $\overrightarrow{OA}$. For simplicity we can chose $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\overrightarrow{OA} \perp \overrightarrow{PA}$, $\therefore \overrightarrow{OA} \cdot \overrightarrow{PA} = 0$. This leave A and B as the possible answers. From $P$, a displacement vector of $\begin{pmatrix} -7 \\ 7 \end{pmatrix}$ would get to $A$ in the first quadrant.

**9** Which of the following integrals correctly represents the area of the shaded region below?




A. $\int_{-a}^{a} \left( x^2 - \frac{1}{2}\cos 4x - \frac{1}{2} \right) dx$

B. $\int_{-a}^{a} \left( -x^2 + \frac{1}{2}\cos 2x + \frac{1}{2} \right) dx$

C. $2\int_{0}^{a} \left( -x^2 + \frac{1}{2}\cos 4x + \frac{1}{2} \right) dx$

D. $2\int_{0}^{a} \left( -x^2 + \frac{1}{2}\cos 4x - \frac{1}{2} \right) dx$

$$A = \int_{-a}^{a} \left[ \left( -x^2 + 1 \right) - \sin^2 2x \right] dx$$

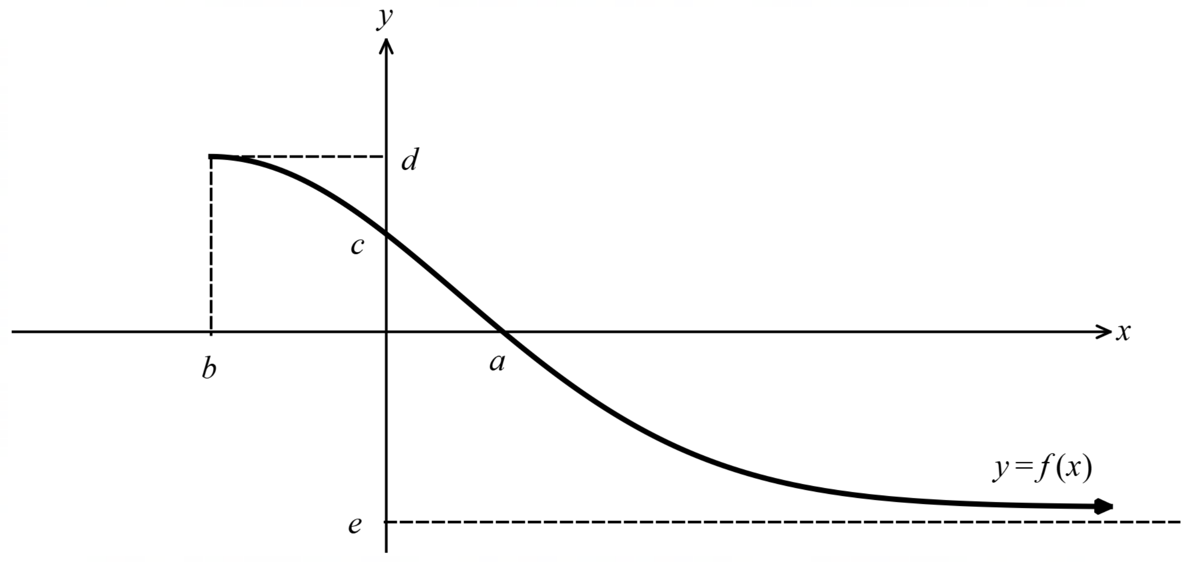
$$= 2\int_{0}^{a} \left[ \left( -x^2 + 1 \right) - \sin^2 2x \right] dx \because \text{ the functions are even}$$

$$= 2\int_{0}^{a} \left[ -x^2 + 1 - \frac{1}{2}\left( 1 - \cos 4x \right) \right] dx$$

$$= 2\int_{0}^{a} \left( -x^2 + 1 - \frac{1}{2} + \frac{1}{2}\cos 4x \right) dx$$

$$= 2\int_{0}^{a} \left( -x^2 + \frac{1}{2}\cos 4x + \frac{1}{2} \right) dx$$

**10** The graph of $y = f(x)$ is given below.




The function $g(x)$ is defined as $g(x) = f^{-1}(x)$. Which of the following gives the range of $g(|x|)$?

A. $[b, \infty)$

B. $[b, a]$

C. $[c, d]$

D. $(e, d]$

By considering the graph of $y = f^{-1}(x)$, range of $f^{-1}(x): y \geq b$

Then by considering the graph of $y = f^{-1}(|x|)$, range of $f^{-1}(|x|): b \leq y \leq a$

**End of Section I**

## Section II

**60 marks**
**Attempt Questions 11-14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

---

**Question 11** (15 marks) Use a SEPARATE writing booklet

(a) For the vectors $\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, evaluate each of the following.

(i) $\underset{\sim}{u} - 2\underset{\sim}{v}$ **1**

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} - 2\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$$

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 2 \times (-3) + (-1) \times 4 = -10$$

(b) Solve $\dfrac{4-x}{x-1} \le 6$. **3**

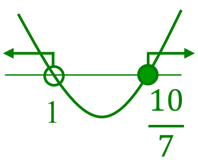
$$\frac{4-x}{x-1} \times (x-1)^2 \le 6 \times (x-1)^2$$

$$(4-x)(x-1) \le 6(x-1)^2$$

$$6(x-1)^2 - (x-1)(4-x) \ge 0$$

$$(x-1)\left[6(x-1) - (4-x)\right] \ge 0$$

$$(x-1)(7x-10) \ge 0$$




$$\therefore x < 1, x \ge \frac{10}{7} \quad \because x \ne 1$$

(c) By using the compound angle formulae, find the exact value of $\tan 15°$. **3**

$$\tan(45°-30°)=\frac{\tan 45°-\tan 30°}{1+\tan 45° \tan 30°}$$

$$\tan 15°=\frac{1-\frac{1}{\sqrt{3}}}{1+1\times\frac{1}{\sqrt{3}}}$$

$$\tan 15°=\frac{1-\frac{1}{\sqrt{3}}}{1+1\times\frac{1}{\sqrt{3}}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$\tan 15°=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(d) Eight people are to be seated at a round table.

(i) How many seating arrangements are possible? **1**

$$7!$$

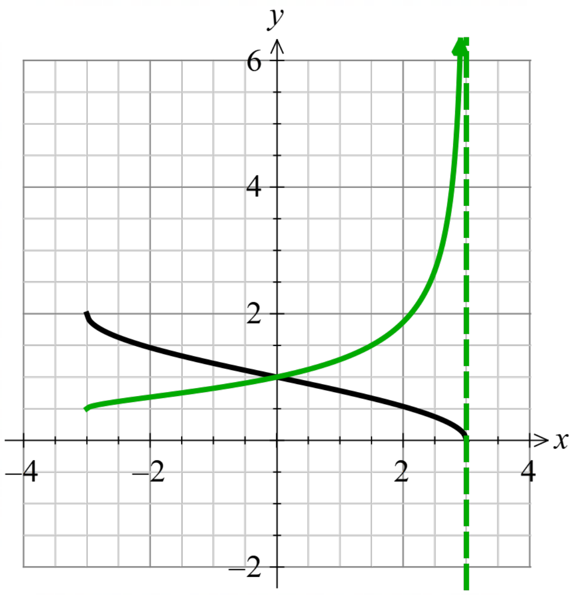
(ii) Patrick wishes to sit between two of his friends, Jack and Holly. **2**
How many seating arrangements are now possible?

$$5!\times 2!$$

**Question 11 continues on page 9**

**<u>Answer Question 11 (e) on the separate Response Sheet.</u>**

(e) The graph of $y = f(x)$ is shown on the separate Response Sheet for Question 11 (e).




(i) Determine the equation of $f(x)$ given that it is in the form $f(x) = a\cos^{-1} bx$. Write your answer below. **2**

$$f(x) = \frac{2}{\pi}\cos^{-1}\frac{x}{3}$$

(ii) On the same grid above, sketch the graph of $y = \dfrac{1}{f(x)}$, clearly showing all important features. **2**

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet

(a) Find $\int (1+\cos\theta)^2\, d\theta$. **3**

$$\begin{aligned}&\int (1+\cos\theta)^2\, d\theta\\ &=\int \left(1+2\cos\theta+\cos^2\theta\right)d\theta\\ &=\int \left(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta\\ &=\int \left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta\\ &=\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta+C\end{aligned}$$

(b) (i) Find the coefficient of $x^k$ in the expansion of $(2+3x)^6$. **1**

$$(2+3x)^6=\binom{6}{0}2^6+\binom{6}{1}2^5\times(3x)^1+\ldots+\binom{6}{k}2^{6-k}\times(3x)^k+\ldots+\binom{6}{6}(3x)^6$$

Therefore, the coefficient of $x^k$ is $\binom{6}{k}3^k\times 2^{6-k}$.

(ii) Hence, or otherwise, find the coefficient of $x^3$ in the expansion of $(2+3x)^6(1-4x^2)$. **2**

coefficient of $x^3$ can be obtained in two ways:

$x^3$ from $(2+3x)^6$ times 1: $\binom{6}{3}3^3\times 2^{6-3}x^3\times 1=4320x^3$

$x^1$ from $(2+3x)^6$ times $-4x^2$ : $\binom{6}{1}3^1\times 2^{6-1}x^1\times -4x^2=-2304x^3$

Therefore, the coefficient of $x^3$ in the expansion is: $4320-2304=2016$

(c) A curve is defined in parametric form by the equations $x = \sqrt{t+1}$ and $y = \ln t$.

(i) Write the equation of the curve in Cartesian form. **1**

$$t = x^2 - 1$$
$$y = \ln\left(x^2 - 1\right)$$

(ii) Sketch the curve clearly, showing all important features. **2**

From the parametric equations:

$$y = \ln t$$
$$\therefore t > 0$$

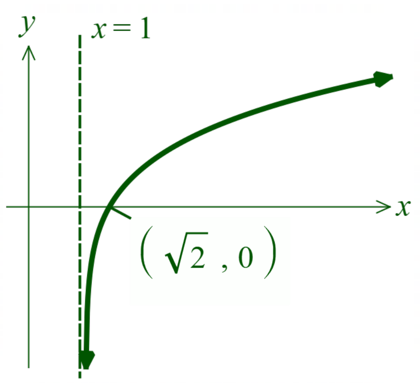
$$x = \sqrt{t+1}\,,\ t > 0$$
$$\therefore x > 1$$

Alternatively:

$$y = \ln\left(x^2 - 1\right) = \ln\left[(x-1)(x+1)\right] = \ln(x+1) + \ln(x-1)$$
$$\therefore x > 1$$

$x$ intercept:

$$\ln\left(x^2 - 1\right) = 0$$
$$\left(x^2 - 1\right) = 1$$
$$x^2 = 2$$
$$x = \sqrt{2} \quad \because x > 1$$



(d) Use the substitution $u^2 = 9 - x$ to find $\int_0^5 \frac{x}{\sqrt{9-x}}\,dx$. **3**

$$x = 9 - u^2$$

$$\frac{dx}{du} = -2u$$

$$dx = -2u\,du \qquad \text{when } x = 5,\ u = \sqrt{9-5} = 2$$

$$\text{when } x = 0,\ u = \sqrt{9-0} = 3$$

$$\int_0^5 \frac{x}{\sqrt{9-x}}\,dx$$

$$= \int_3^2 \frac{9-u^2}{u}\left(-2u\,du\right)$$

$$= 2\int_2^3 9 - u^2 du$$

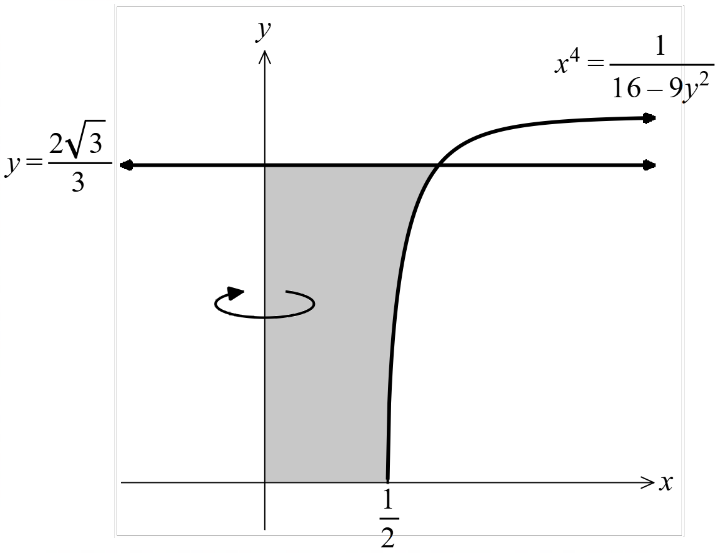
$$= 2\left[9u - \frac{u^3}{3}\right]_2^3$$

$$= 2\left[\left(9\times 3 - \frac{3^3}{3}\right) - \left(9\times 2 - \frac{2^3}{3}\right)\right]$$

$$= \frac{16}{3}$$

**Question 12 continues on page 11**

(e) A concrete block is formed by rotating the shaded region bounded by the graph of $x^4 = \dfrac{1}{16-9y^2}$, the $x$ axis, the $y$ axis and the line $y = \dfrac{2\sqrt{3}}{3}$ about the $y$ axis. **3**




Find the exact volume of the solid formed.

$$x^2 = \frac{1}{\sqrt{16-9y^2}}$$

$$V = \pi\int_0^{\frac{2\sqrt{3}}{3}} x^2\, dy$$

$$= \pi\int_0^{\frac{2\sqrt{3}}{3}} \frac{1}{\sqrt{16-9y^2}}\, dy$$

$$= \pi\int_0^{\frac{2\sqrt{3}}{3}} \frac{1}{\sqrt{4^2-(3y)^2}}\, dy$$

$$= \frac{\pi}{3}\int_0^{\frac{2\sqrt{3}}{3}} \frac{3}{\sqrt{4^2-(3y)^2}}\, dy$$

$$= \frac{\pi}{3}\left[\sin^{-1}\frac{3y}{4}\right]_0^{\frac{2\sqrt{3}}{3}}$$

$$= \frac{\pi}{3}\left(\sin^{-1}\left(\frac{3}{4}\times\frac{2\sqrt{3}}{3}\right) - \sin^{-1}\left(\frac{3}{4}\times 0\right)\right)$$

$$= \frac{\pi}{3}\sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3}\times\frac{\pi}{3}$$

$$= \frac{\pi^2}{9}\, u^3$$

**Question 13** (16 marks) Use a SEPARATE writing booklet

(a) Let $f(x)=\sin^{-1}x+\cos^{-1}x$.

(i) By differentiating $f(x)$, or otherwise, explain why $f(x)=\frac{\pi}{2}$. **2**

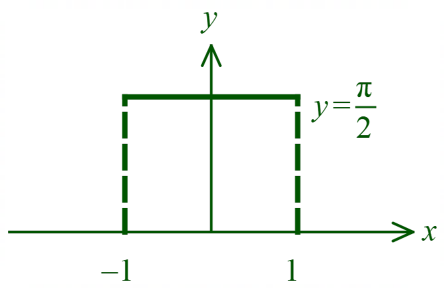
$$f'(x)=\frac{1}{\sqrt{1-x^2}}+\left(-\frac{1}{\sqrt{1-x^2}}\right)=0$$

Since $f'(x)=0$, then $f(x)$ must be equal to a constant. Substituting $x=0$ into $f(x)$ gives:

$$f(0)=\sin^{-1}0+\cos^{-1}0=0+\frac{\pi}{2}=\frac{\pi}{2}$$

(ii) Hence, sketch the graph of $y=f(x)$. **1**

Domain of $f(x)$: $-1\le x\le 1$



(b) Let $f(x)=18x^2+3px-2q$ and $g(x)=18x^2+3qx-2p$, where $p$, $q$ are distinct real numbers. $\alpha,\beta$ are the roots of the equation $f(x)=0$ and $\alpha,\gamma$ are the roots of the equation $g(x)=0$.

(i) By first finding the value of $\alpha$, show that $p+q=4$. **3**

$$f(\alpha)=g(\alpha)=0$$
$$18\alpha^2+3p\alpha-2q=18\alpha^2+3q\alpha-2p$$
$$3(p-q)\alpha=-2(p-q)$$
$$3(p-q)\alpha+2(p-q)=0$$
$$(p-q)(3\alpha+2)=0$$
$$\alpha=\frac{-2}{3}\ (\because p\neq q)$$

$$f(\alpha)=0$$
$$18\left(\frac{-2}{3}\right)^2+3p\left(\frac{-2}{3}\right)-2q=0$$
$$8-2p-2q=0$$
$$2(p+q)=8$$
$$p+q=4$$

(ii) Hence, or otherwise, express $\beta$ and $\gamma$ in terms of $p$. **2**

Consider the sum of roots of $f(x)$ and $g(x)$

$$\alpha+\beta=\frac{-3p}{18}$$
$$\beta=-\frac{p}{6}-\alpha$$
$$\beta=-\frac{p}{6}+\frac{2}{3}$$
$$\beta=\frac{4-p}{6}$$

$$\alpha+\gamma=\frac{-3q}{18}$$
$$\gamma=-\frac{q}{6}-\alpha$$
$$\gamma=-\frac{q}{6}+\frac{2}{3}$$
$$\gamma=\frac{4-q}{6}$$
$$\gamma=\frac{4-(4-p)}{6}$$
$$\gamma=\frac{p}{6}$$

(c) Use Mathematical Induction to prove that $3^{3n}+2^{n+2}$ is divisible by 5 for all integers $n \geq 1$. **3**

when $n=1$,

$$3^{3\times1}+2^{1+2}=3^3+2^3=27+8=35=5\times7$$

$\therefore$ the statement is true for $n=1$.

assume that it is true for $n=k$, $k \in Z^+$

$$3^{3k}+2^{k+2}=5A,\ A\in Z^+$$

$$\therefore 3^{3k}=5A-2^{k+2}$$

when $n=k+1$,

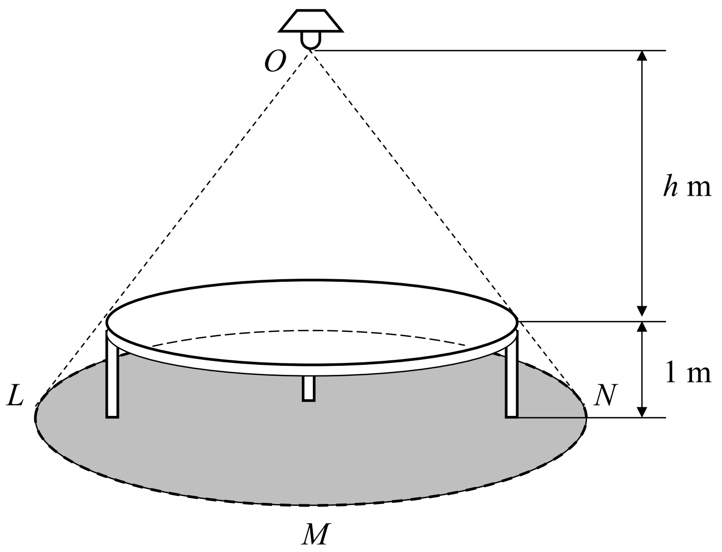
$$\begin{aligned}
&3^{3(k+1)}+2^{(k+1)+2}\\
&=3^{3k+3}+2^{k+3}\\
&=3^3\times3^{3k}+2^{k+3}\\
&=3^3\times\left(5A-2^{k+2}\right)+2^{k+3}\ (\text{by assumption})\\
&=5\times3^3A-3^3\times2^{k+2}+2\times2^{k+2}\\
&=5\times3^3A-2^{k+2}\left(3^3-2\right)\\
&=5\times3^3A-2^{k+2}\times25\\
&=5\left(3^3A-2^{k+2}\times5\right)\\
&=5B\quad \text{where } B=\left(3^3A-2^{k+2}\times5\right)\in Z, \because A\in Z^+
\end{aligned}$$

Hence, $3^{3(k+1)}+2^{(k+1)+2}$ is divisible by 5 if $3^{3k}+2^{k+2}$ is divisible by 5.
Therefore, the statement is true by mathematical induction.

**Question 13 continues on page 13**

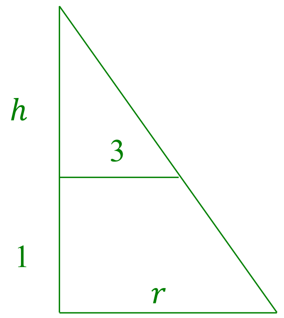
(d) A small lamp $O$ is placed $h$ m above the centre of a round table of radius 3 m and height of 1 m. The lamp casts a circular shadow $LMN$ of the table on the ground as shown in the diagram below.

At any given time, let $h$ be the height, in metres, of the lamp above the table and let $A$ be the area of the shadow in $\text{m}^2$.




(i) Show that $A = 9\pi\left(1+\frac{1}{h}\right)^2$. **2**

By considering similiar triangles:




$$\frac{h+1}{h} = \frac{r}{3}$$

$$r = 3\left(\frac{h+1}{h}\right)$$

$$r = 3\left(1+\frac{1}{h}\right)$$

$$A = \pi r^2$$

$$A = \pi\left(3\left(1+\frac{1}{h}\right)\right)^2$$

$$A = 9\pi\left(1+\frac{1}{h}\right)^2$$

(ii) If the lamp is lowered vertically at a constant rate of $\frac{3}{100}$ m/s, find the exact rate of change of $A$ with respect to time when the lamp is 3 m above the table. **3**

$$A = 9\pi\left(1+\frac{1}{h}\right)^2$$

$$\frac{dA}{dh} = 9\pi \times 2\left(1+\frac{1}{h}\right)\times\left(-\frac{1}{h^2}\right) = -\frac{18\pi}{h^2}\left(1+\frac{1}{h}\right)$$

when $A = 16\pi$

$$16\pi = 9\pi\left(1+\frac{1}{h}\right)^2$$

$$1+\frac{1}{h} = \pm\frac{4}{3}$$

$$\frac{1}{h} = \frac{1}{3} \quad \because h \geq 0$$

$$h = 3$$

when $h = 3$

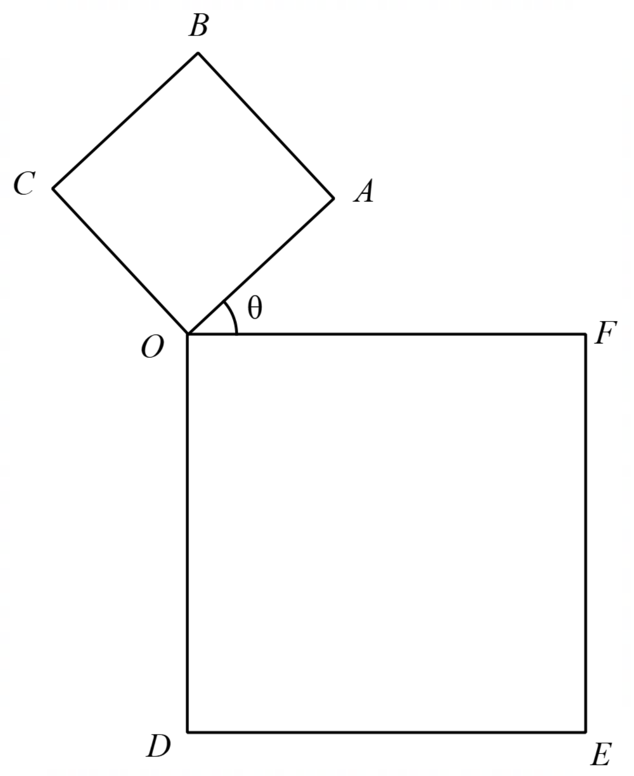
$$\frac{dA}{dh} = -\frac{18\pi}{3^2}\left(1+\frac{1}{3}\right) = \frac{-8\pi}{3}$$

$$\frac{dA}{dt} = \frac{dA}{dh}\times\frac{dh}{dt} = \frac{-8\pi}{3}\times\frac{-3}{100} = \frac{2\pi}{25}\ \text{m}^2/\text{s}$$

**End of Question 13**

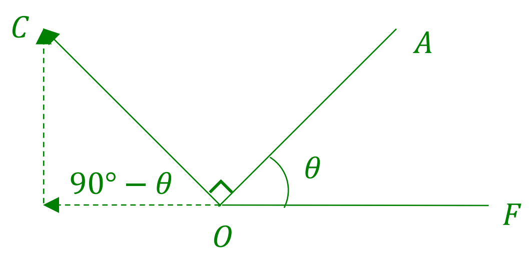
**Question 14** (14 marks) Use a SEPARATE writing booklet

(a) The diagram below shows two squares $OABC$ and $ODEF$ with $OA = 1$, $OD = 2$ and $\angle AOF = \theta$ , where $0° < \theta < 90°$. Let $\overrightarrow{OF} = 2\underset{\sim}{i}$, $\overrightarrow{OD} = -2\underset{\sim}{j}$ and $\overrightarrow{OA} = \cos\theta\ \underset{\sim}{i} + \sin\theta\ \underset{\sim}{j}$.




(i) Explain why $\overrightarrow{OC} = -\sin\theta\ \underset{\sim}{i} + \cos\theta\ \underset{\sim}{j}$. **1**

From the diagram below, $\overrightarrow{OC}$ makes an angle of $90° - \theta$ with the horizontal.




By resolving $\overrightarrow{OC}$ into components, we get:

$$\overrightarrow{OC} = -\cos(90° - \theta)\ \underset{\sim}{i} + \sin(90° - \theta)\ \underset{\sim}{j}$$

$$\overrightarrow{OC} = -\sin\theta\ \underset{\sim}{i} + \cos\theta\ \underset{\sim}{j}$$

(ii) Use vector methods to show that if the points $B$, $A$ and $E$ are collinear, **3**

$\cos\theta - \sin\theta = \frac{1}{2}$.

$$\overrightarrow{BA} = -\overrightarrow{OC} = -\left(-\sin\theta\, \underset{\sim}{i} + \cos\theta\, \underset{\sim}{j}\right) = \sin\theta\, \underset{\sim}{i} - \cos\theta\, \underset{\sim}{j}$$

$$\overrightarrow{AE} = -\overrightarrow{OA} + \overrightarrow{OD} + \overrightarrow{DE} = -\left(\cos\theta\, \underset{\sim}{i} + \sin\theta\, \underset{\sim}{j}\right) + \left(-2\underset{\sim}{j}\right) + 2\underset{\sim}{i} = \left(2 - \cos\theta\right)\underset{\sim}{i} + \left(-2 - \sin\theta\right)\underset{\sim}{j}$$

if $B, A, E$ are collinear, then $\overrightarrow{BA} \parallel \overrightarrow{AE}$

$$\overrightarrow{BA} = \lambda \overrightarrow{AE}$$

$$\sin\theta = \lambda\left(2 - \cos\theta\right)$$
$$-\cos\theta = \lambda\left(-2 - \sin\theta\right)$$

$$\frac{\sin\theta}{2 - \cos\theta} = \frac{-\cos\theta}{-2 - \sin\theta}$$
$$\left(2 + \sin\theta\right)\sin\theta = \cos\theta\left(2 - \cos\theta\right)$$
$$2\sin\theta + \sin^2\theta = 2\cos\theta - \cos^2\theta$$
$$\sin^2\theta + \cos^2\theta = 2\cos\theta - 2\sin\theta$$
$$1 = 2\cos\theta - 2\sin\theta$$
$$\cos\theta - \sin\theta = \frac{1}{2}$$

(iii) By first expressing $\cos\theta - \sin\theta$ in the form of $R\cos(\theta+\alpha)$ where $R>0$ and $0<\alpha<90°$, find the value(s) of $\theta$ such that points $B$, $A$ and $E$ are collinear. Give your answer(s) correct to the nearest degree. Show all working. **3**

$$\cos\theta - \sin\theta = R\cos(\theta+\alpha)$$
$$\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$
$$\cos\theta - \sin\theta = (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$$

By comparing coefficients of $\cos\theta$ and $\sin\theta$ on both sides:

$$R\cos\alpha = 1$$
$$R\sin\alpha = 1$$

Solving simultaneouly gives:

$$R^2 = 2$$
$$R = \sqrt{2} \quad (\because R>0)$$

$$\tan\alpha = 1$$
$$\alpha = 45° \quad (\because 0° < \alpha < 90°)$$

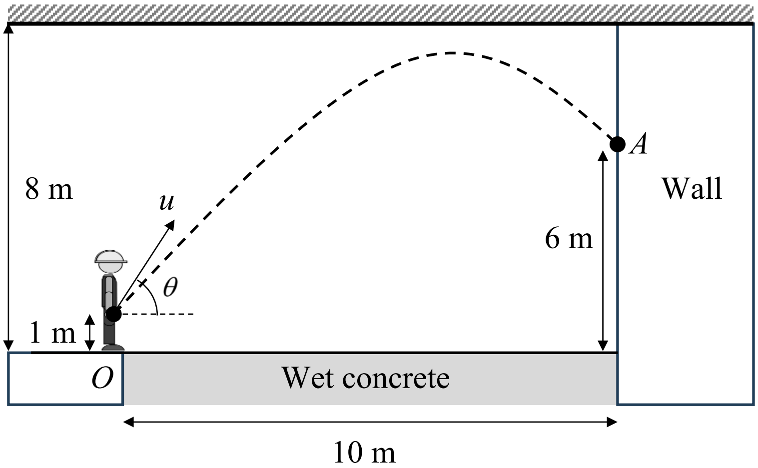
$$\therefore \cos\theta - \sin\theta = \sqrt{2}\cos(\theta + 45°)$$

$$\sqrt{2}\cos(\theta+45°) = \frac{1}{2}$$
$$\cos(\theta+45°) = \frac{1}{2\sqrt{2}}$$
$$\theta + 45° = \cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$
$$\theta + 45° = 69°, 291°$$
$$\theta = 24° \quad (\because 0° < \theta < 90°)$$

**Question 14 continues on page 15**

(b) A construction worker wants to anchor a cable at point $A$ on a wall inside a room. The room is 8 m high and the anchor point for the cable at $A$ is 6 m above the ground on the wall. A 10 m patch of wet concrete prevents the worker from getting close to the wall as shown in the diagram below.

In order to anchor the cable to the wall, the worker needs to launch the cable hook from his hand 1 m above ground at a speed of $u$ m/s and an angle of $\theta°$ to the horizontal. The acceleration due to gravity is $g$ m/s$^2$.




Taking the position of the worker's foot as the origin, the position of the cable hook at time $t$ seconds after it is launched is given by:

$$\vec{r} = \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta + 1 \end{pmatrix}$$ (Do NOT prove this)

(i) Show that for the cable hook to avoid hitting the ceiling, $\sin^2\theta < \dfrac{14g}{u^2}$. **3**

from the vector equation:

$$\dot{\vec{r}} = \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix}$$

at max height,

$$-gt + u\sin\theta = 0$$

$$t = \frac{u\sin\theta}{g}$$

max height reached by the projectile is given by

$$-\frac{g\left(\dfrac{u\sin\theta}{g}\right)^2}{2} + u\left(\frac{u\sin\theta}{g}\right)\sin\theta + 1$$

$$= -\frac{u^2\sin^2\theta}{2g} + \frac{u^2\sin^2\theta}{g} + 1$$

$$= \frac{u^2\sin^2\theta}{2g} + 1$$

in order for the cable hook to reach the wall without hitting the ceiling

$$\frac{u^2\sin^2\theta}{2g} + 1 < 8$$

$$\frac{u^2\sin^2\theta}{2g} < 7$$

$$\sin^2\theta < \frac{14g}{u^2} \quad \because \frac{u^2}{2g} > 0$$

The worker launches the cable hook at 15 m/s. Assuming that $g = 10\ \text{m/s}^2$, the Cartesian equation of the cable hook is given by:

$$y = -\frac{x^2}{45}\sec^2\theta + x\tan\theta + 1 \qquad \text{(Do NOT prove this)}$$

(ii) Find all possible angle(s) that the worker can launch the cable hook to reach the anchor point. Round your answer(s) to the nearest degree. **4**

sub $x = 10$, $y = 6$

$$6 = -\frac{10^2}{45}\sec^2\theta + 10\tan\theta + 1$$

$$6 = -\frac{10^2}{45}\left(1 + \tan^2\theta\right) + 10\tan\theta + 1$$

$$270 = -100\left(1 + \tan^2\theta\right) + 450\tan\theta + 45$$

$$270 = -100 - 100\tan^2\theta + 450\tan\theta + 45$$

$$100\tan^2\theta - 450\tan\theta + 325 = 0$$

$$4\tan^2\theta - 18\tan\theta + 13 = 0$$

Using the quadratic formula to solve for $\tan\theta$ gives:

$\tan\theta = 3.5962...$ $\qquad$ $\tan\theta = 0.9037...$

$\theta = 74°$ $\qquad$ $\theta = 42°$

$$\frac{14g}{u^2} = \frac{14\times 10}{15^2} \approx 0.62$$

when $\theta = 42°, \sin^2 42° \approx 0.45 < 0.61 \therefore$ it will not hit the ceiling

when $\theta = 74°, \sin^2 74° \approx 0.92 > 0.61 \therefore$ it will hit the ceiling

$\therefore$ the worker has to launch the cable hook at $42°$

**End of paper**